TUTORIA - 1.

1. A RV X has the following prob. distribution.

x	:	0	1	2	3	4	5	6	7
$P\{X=x\}$	:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$.

i) Find k.

$$\sum p_i = 1$$

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$10k^2+9k = 1$$

$$10k^2+9k-1 = 0.$$

$$(k+1)(10k-1) = 0.$$

$k+1=0$ or $10k-1=0$.

$k=-1$ (not possible) $\quad k=\frac{1}{10}$.

$-10 < \frac{10}{10}, \frac{-1}{10}$, $\underline{9}$

x	:	0	1	2	3	4	5	6	7
$P(X=x)$	:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

ii) Find $P(X<6)$, $P(X\geq 6)$, $P(0<x<5)$

$$P(X\geq 6) = P(X=6)+P(X=7)$$

$$= \frac{2}{100}+\frac{17}{100}$$

$$= \frac{19}{100}$$

$$P(X<6) = 1-P(X\geq 6)$$

$$= 1-\frac{19}{100}$$

$$= \frac{100-19}{100}$$

$$= \frac{81}{100}$$

iii) $P(0<x<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$$= \frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}$$

$$= \frac{8}{10}$$

iii) Find CDF

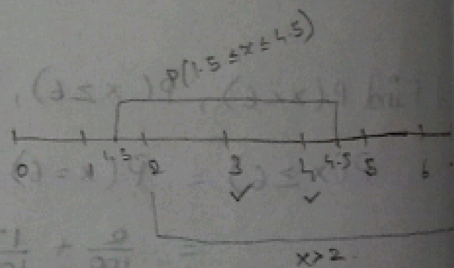
x	$f(x)$	$F(x) = P(X \le x)$
0	0	0
1	$\frac{1}{10}$	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{3}{10}$
3	$\frac{2}{10}$	$\frac{5}{10} = \frac{1}{2}$
4	$\frac{3}{10}$	$\frac{8}{10} > \frac{1}{2}$
5	$\frac{1}{100}$	$\frac{81}{100} > \frac{1}{2}$
6	$\frac{2}{100}$	$\frac{83}{100}$
7	$\frac{17}{100}$	$\frac{100}{100} = 1$

iv) Find the minimum value of $k \ni P(X \le k) > \frac{1}{2}$.

w.k.t. $F(x) = P(X \le x)$

$$P(X \le k) > \frac{1}{2} = 4$$

v) $P(1.5 \le x \le 4.5 / x > 2)$



$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(1.5 \le x \le 4.5 / x > 2) = \frac{P(1.5, 4.5)}{}$$

$$= \frac{P(1.5 \le x \le 4.5,\ x > 2)}{P(x > 2)}$$

$$= \frac{P(x=3) + P(x=4)}{P(x>2)}$$

$$= \frac{2/10 + 3/10}{1 - P(x \le 2)}$$

$$= \frac{5/10}{1 - [P(x=0) + P(x=1) + P(x=2)]}$$

$$= \frac{5/10}{1 - [0 + \frac{1}{10} + \frac{2}{10}]} = \frac{5/10}{\frac{10-3}{10}} = \frac{5}{7}.$$

2. A RV, X takes the values 1, 2, 3, 4, 5
$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the prob. distribution

Sol: Given

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$$

Let, $P(X=3) = k$

$$2P(X=1) = k \Rightarrow P(X=1) = \frac{k}{2}$$

$$3P(X=2) = k \Rightarrow P(X=2) = \frac{k}{3}$$

$$5P(X=4) = k \Rightarrow P(X=4) = \frac{k}{5}$$

To find k:

$$\Sigma p_i = 1$$

$$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{5} = 1$$

$$\frac{30k + 15k + 10k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$k = \frac{30}{61}$$

x :	1	2	3	4
$P(X=x)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

CDF :

x	$f(x)$	$F(x) = P(X \le x)$
1	$\frac{15}{61}$	$\frac{15}{61}$
2	$\frac{10}{61}$	$\frac{25}{61}$
3	$\frac{30}{61}$	$\frac{55}{61}$
4	$\frac{6}{61}$	$\frac{61}{61} = 1$

$$\text{Mean} = \Sigma x_i p_i = 1 \cdot \frac{15}{61} + 2 \cdot \frac{10}{61} + 3 \cdot \frac{30}{61} + 4 \cdot \frac{6}{61}$$

$$= \frac{15}{61} + \frac{20}{61} + \frac{90}{61} + \frac{24}{61}$$

$$= \frac{149}{61}$$

$$\text{Variance} = \sum x_i^2 p_i = 1^2 \cdot \frac{15}{61} + 2^2 \cdot \frac{10}{61} + 3^2 \cdot \frac{30}{61} + 4^2 \cdot \frac{6}{61}$$

$$= \frac{15}{61} + \frac{40}{61} + \frac{270}{61} + \frac{96}{61}$$

$$= \frac{421}{61}.$$

3. Prob function of an infinite discrete distribution is

$P(x=j) = \frac{1}{2^j}, \; j = 1, 2, \ldots$

Sol: Given, $P(x=j) = \frac{1}{2^j}, \; j = 1, 2, \ldots$

$x=j$:	1	2	3	4	....
$P(x=j)$:	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	....

i) $P\{x = \text{even}\} = P(x=2) + P(x=4) + P(x=6 + \ldots$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots$$

$$= \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right]$$

$$= \frac{1}{2^2}\left[1 + \left(\frac{1}{2^2}\right) + \left(\frac{1}{2^2}\right)^2 + \ldots\right] \qquad 1 + x + x^2 + \ldots =$$

$$= \frac{1}{2^2}(1-x)^{-1} \qquad x = \frac{1}{2^2}$$

$$= \frac{1}{4}\left(1 - \frac{1}{4}\right)^{-1}$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^{-1}$$

$$= \frac{\frac{1}{4}}{3/4}$$

$$= \frac{1}{3}$$

ii) $P(x \geq 5) = 1 - P(x < 5)$

$$= 1 - [P(x=1) + P(x=2) + P(x=3) + P(x=4)]$$

$$= 1 - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right]$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)$$

$$= 1 - \left[\frac{8+4+2+1}{16}\right]$$

$$= 1 - \frac{15}{16}$$

$$P(X \geq 5) = \frac{1}{16}.$$

iii) $P\{X \div \text{by } 3\}$. $= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots$

$$= \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \dots\right]$$

$$= \frac{1}{8}\left[1 + \left(\frac{1}{2^3}\right) + \left(\frac{1}{2^3}\right)^2 + \dots\right]$$

$1 + x + x^2 + \dots = (1-x)^{-1}$

$x = \frac{1}{2^3}$

$$= \frac{1}{8}\left[1 - \frac{1}{2^3}\right]^{-1}$$

$$= \frac{1}{8}\left(1 - \frac{1}{8}\right)^{-1} = \frac{1}{8}\left(\frac{7}{8}\right)^{-1}$$

$$= \frac{1}{8} \times \frac{8}{7} = \frac{1}{7}$$

iv) Find MGF

W.k.t $\quad M_X(t) = E[e^{tj}] = \sum e^{tj} P(X=j)$

$$= \sum_{j=1}^{\infty} e^{tj} \cdot \frac{1}{2^j} = \sum_{j=1}^{\infty} \left(\frac{e^t}{2}\right)^j$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \dots\right]$$

$= \frac{e^t}{2}(1 + x + x^2 + \dots)$, where $x = \frac{e^t}{2}$

$$= \frac{e^t}{2}(1-x)^{-1}$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2} \times \frac{2}{2-e^t}$$

$$M_X(t) = \frac{e^t}{2-e^t}$$

4. The probability function of an infinite discrete distribution is given by $P(x=j)=\frac{1}{2^j}$, $j=1,2,3,\ldots$. Find the mean and variance of the distribution. Also find $P(x=\text{even})$, $P(x \text{ is} \div \text{by } 3)$, $P(x \geq 5)$.

Sol: Given $P(x=j)=\frac{1}{2^j}$, $j=1,2,\ldots$

$x=j$:	1	2	3	4	5	$\ldots$
$P(x=j)$:	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	$\ldots$

Mean:

$$E(x)=\sum x\,P(x=x)$$

$$=1\cdot\frac{1}{2}+2\cdot\frac{1}{2^2}+3\cdot\frac{1}{2^3}+4\cdot\frac{1}{2^4}+\ldots$$

$$=\frac{1}{2}+2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+4\left(\frac{1}{2}\right)^4+\ldots$$

$$=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots\right]$$

$1+2x+3x^2+4x^3+\ldots=(1-x)^{-2}$

$$=\frac{1}{2}\left[1+\boxed{2}x+3x^2+4x^3+\ldots\right], \text{ where } x=\frac{1}{2}$$

$$=\frac{1}{2}(1-x)^{-2}=\frac{1}{2}\left(1-\frac{1}{2}\right)^{-2}$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)^{-2}=\frac{1}{2}\times 2^2=\frac{\cancel{4}^{2}}{\cancel{2}}=2$$

$$E(x)=2$$

To find $E(x^2)$:

$$E(x^2)=\sum x^2 p(x=x)=\sum\left(x(x-1)+x\right)P(x=x)$$

$$=\sum_{x=1}^{\infty} x(x-1)P(x=x)+\underbrace{\sum x\,P(x=x)}_{E(x)}$$

$$= \left[0 + 2\cdot 1 \cdot P(x=2) + 3\cdot 2\, P(x=3) + 4\cdot 3\cdot P(x=4) + \dots\right] + E(x)$$

$$= \left[2\,\frac{1}{2^2} + 3\cdot 2\cdot \frac{1}{2^3} + 4\cdot 3\cdot \frac{1}{2^4} + \dots\right] + 2$$

$$= \left[2\left(\frac{1}{2}\right)^2 + 2\cdot 3\left(\frac{1}{2}\right)^3 + 3\cdot 4\left(\frac{1}{2}\right)^4 + \dots\right] + 2$$

$$= 2\left(\frac{1}{2}\right)^2\left[1 + \boxed{3}\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + \dots\right] + 2$$

$$= \frac{1}{2}(1-x)^{-3} + 2$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \dots$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right)^{-3} + 2$$

$$(1-x)^{-3} = 1 + 3x + \frac{3\cdot 4}{2}x^2 + \dots$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-3} + 2$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \dots$$

$$= \frac{1}{\cancel{2}}\left(2^{\cancel{3}}\right)^{2} + 2 \quad = 2^2 + 2$$

$$E(x^2) = 6$$

Variance. $V(x) = E(x^2) - (E(x))^2$

$$= 6 - 2^2$$

$$V(x) = 2$$

2. A random variable x has a density function $f(x) = \begin{cases} \frac{k}{1+x^2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

i) find k, ii) find $P(x>0)$ iii) find CDF.

Sol. W.k.t

i)
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\tan^{-1}\infty = \pi/2$$

$$\tan^{-1} 0 = 1$$

$$k\left(\tan^{-1}x\right)_0^{\infty} = 1$$

$$k\left(\frac{\pi}{2} - 0\right) = 1$$

$$k = \frac{2}{\pi}$$

$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

ii) $P(x>0)$
$$= \int_0^{\infty} f(x)\,dx = \int_0^{\infty} \frac{2}{\pi(1+x^2)}\,dx = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \frac{2}{\pi}\left(\tan^{-1}x\right)_0^{\infty} = \cancel{\pi}\ \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{2}{\pi}\left(\frac{\pi}{2}\right)$$